COPULÆ AND THEIR USES

Carlo Sempi
Dipartimento di Matematica "Ennio De Giorgi"
Università di Lecce
I–73100 Lecce, Italy
E-mail: carlo.sempi@unile.it

This survey of copulas reviews some aspects of copulas, their properties, tries to stress their relevance for statistics and their connection with Markov processes and conditional expectations.

1. What is a Copula?

Copulæ were introduced in 1959 by Sklar[25]. Nowadays the literature on copulas is very large. The reader is referred in the first place to the authoritative books and surveys by Schweizer, Sklar and Nelsen[26,21,20,27,15]. Also the books by Joe[13] and by Hutchinson and Lai[12] contain much useful information; so do also a wealth of papers, some of which will be cited when the need arises. The present paper aims at introducing some of the properties and of the uses of copulas, even of those not (yet) of immediate relevance for the field of reliability; of course, it cannot pretend to have either the same breadth or the same depth of the works just cited.

A copula is a function $C : [0,1] \times [0,1] \to [0,1]$ that satisfies the following properties:

- for all t in $[0,1]$, $C(t,0) = C(0,t) = 0$;
- for all t in $[0,1]$, $C(t,1) = C(1,t) = t$;
- if x, x', y, y' are in $[0,1]$ with $x \le x'$ and $y \le y'$, then

$$C(x',y') - C(x,y') - C(x',y) + C(x,y) \ge 0. \tag{1}$$

As a consequence of these properties it follows that

(a) C satisfies the Lipschitz condition

$$|C(x,y) - C(x',y')| \le |x - x'| + |y - y'|,$$

(b) it is non–decreasing in each variable,
(c) it is absolutely continuous.

In other words, a copula is (the restriction of) an absolutely continuous bivariate distribution function that concentrates all the probability mass on the unit square $[0,1] \times [0,1]$ and which has uniform marginals. We consider random variables that may take the values $-\infty$ and/or $+\infty$ with probability different from zero. As a consequence, their distribution functions are defined on $\overline{\mathbf{R}} := [-\infty, +\infty]$, in the univariate case, and on $\overline{\mathbf{R}}^2 = \overline{\mathbf{R}} \times \overline{\mathbf{R}}$, in the bivariate case; also they may have a jump at points with at least an infinite coordinate.

The importance of the concept of copula stems from the following

Theorem 1: (Sklar) *Let X and Y be two random variables on the probability space $(\Omega, \mathcal{F}, P)$ having H as their joint distribution function and let F and G be the marginals of H,*

$$F(x) = H(x, +\infty), \qquad G(y) = H(+\infty, y).$$

Then, there exists (at least) a copula C such that

$$H(x,y) = C\left(F(x), G(y)\right). \tag{2}$$

If both F and G are continuous, then the copula C is uniquely determined; otherwise, C is uniquely determined on Ran $F \times$Ran G, *where* Ran F *is the image of $\overline{\mathbf{R}}$ under F,* Ran $F := F\left(\overline{\mathbf{R}}\right)$. *Conversely, if C is a copula and F and G are univariate distribution functions, then the function H defined by (2) is a joint distribution function with marginals given by F and G.*

If either F or G, or both, is not continuous, then more than one copula may satisfy (2); all of these coincide on Ran $F \times$Ran G. Notice that the discrete case is important for the applications because empirical versions of copulas are obtained by substituting empirical distributions for F and G. However, by a method of bilinear interpolation, it is always possible to choose a single copula that satisfies (2), see Section 2.3.5 in the book by Nelsen[15]. This method is essential for some of the developments presented in the sequel. More specifically, let C' be any of the copulas that satisfy (2) and which are uniquely determined on Ran $F \times$Ran G; extend it by continuity to the closure of the set Ran $F \times$Ran G. Now let (s,t) be any point in $[0,1] \times [0,1]$ and let s_1 and s_2 be the greatest and the least element, respectively, in $\overline{\text{Ran } F}$—the closure of Ran F—such that $s_1 \le s \le s_2$. Similarly, let t_1 and t_2 be the greatest and the least element, respectively, in $\overline{\text{Ran } G}$ such that $t_1 \le t \le t_2$. If s belongs to $\overline{\text{Ran } F}$, then $s_1 = s = s_2$; and, if t belongs to $\overline{\text{Ran } G}$, then $t_1 = t = t_2$. Then define

$$\lambda_1 := \begin{cases} \dfrac{s - s_1}{s_2 - s_1}, & \text{if } s_1 < s_2, \\ 1, & \text{if } s_1 = s_2; \end{cases}$$

$$\lambda_2 := \begin{cases} \dfrac{t - t_1}{t_2 - t_1}, & \text{if } t_1 < t_2, \\ 1, & \text{if } t_1 = t_2. \end{cases}$$

It is a long, but not hard, task to check that the function defined on $[0,1] \times [0,1]$ by

$$\begin{aligned} C(s,t) := {} & (1-\lambda_1)\,(1-\lambda_2)\,C'(s_1,t_1) + (1-\lambda_1)\,\lambda_2\,C'(s_1,t_2) \\ & + \lambda_1\,(1-\lambda_2)\,C'(s_2,t_1) + \lambda_1\,\lambda_2\,C'(s_2,t_2) \end{aligned}$$

is a copula and that it satisfies (2).

Among the copulas, three are particularly important; they are denoted by W, Π and M and are defined by

$$\begin{aligned} W(s,t) &:= \max\{0, s+t-1\}, \\ \Pi(s,t) &:= s\,t, \\ M(s,t) &:= \min\{s,t\}. \end{aligned}$$

Two continuous random variables X and Y are independent if, and only if, their copula C_{XY} is equal to Π; two continuous random variables X and Y have W as their copula if, and only if, one of them is a strictly decreasing function of the other one, while they have M as their copula if, and only if, one of them is a strictly increasing function of the other one.

Let X and Y be two continuous random variables and let C_{XY} be their copula. If φ and ψ are strictly increasing functions defined on Ran X and Ran Y respectively, then the copula $C_{\varphi\circ X, \psi\circ Y}$ of the random variables $\varphi \circ X$ and $\psi \circ Y$ satisfies

$$C_{\varphi\circ X, \psi\circ Y} = C_{XY}.$$

Thus the copula C_{XY} is invariant under strictly increasing transformations of X and Y. This property has an important consequence: it makes copulas well suited to express the "scale invariant" properties and measures of association for random variables. This aspect will be exploited in section 3.

Moreover, if C is any copula, the following inequalities (the Fréchet bounds[14]) hold for all s and t in $[0,1]$,

$$W(s,t) \leq C(s,t) \leq M(s,t).$$

The properties of the partial derivatives of a copula C are important for what follows. For every $t \in [0,1]$, the function $s \mapsto C(s,t)$ is non-decreasing and, hence, differentiable for almost every t; where the derivative exists, one has

$$0 \leq D_1 C(s,t) := \frac{\partial C(s,t)}{\partial s} \leq 1 \qquad \text{a.e..}$$

Similarly, for every $s \in [0,1]$, the function $t \mapsto C(s,t)$ is non-decreasing and differentiable almost everywhere in I; where the derivative exists, one has

$$0 \leq D_2 C(s,t) := \frac{\partial C(s,t)}{\partial t} \leq 1 \qquad \text{a.e..}$$

2. Special Classes of Copulas

In the literature, one can find several methods of constructing copulas; see, for this, Chapter 3 in the book by Nelsen[15]. The most widely studied class of copulas is probably that of *Archimedean* copulas. The reason for which these copulæ have been so extensively studied is twofold: on one hand, they are symmetric and, therefore, they lend themselves admirably to the study of pairs of exchangeable random variables, and, on the other hand, many of the families presented in the literature depend on one or more parameters, which allows the usual statistical procedures of best estimation and goodness of fit in concrete case studies; see, for instance De Michele and Salvadori[7].

In order to define Archimedean copulas, let $\varphi : [0,1] \to [0,+\infty]$ be continuous, convex, strictly decreasing and such that $\varphi(1) = 0$. The *pseudo-inverse* of φ is the function $\varphi^{[-1]}$ from $[0,+\infty]$ into $[0,1]$ defined by

$$\varphi^{[-1]}(t) := \begin{cases} \varphi^{-1}(t), & t \in [0, \varphi(0)]\,, \\ 0, & t \in [\varphi(0), +\infty]\,. \end{cases}$$

For every $t \in [0,1]$, one has

$$\varphi^{[-1]}(\varphi(t)) = t,$$

while, for every $x \in [0,+\infty]$, one has

$$\varphi\left(\varphi^{[-1]}(x)\right) = \min\{x, \varphi(0)\}.$$

Then, an Archimedean copula C is defined via

$$C(s,t) := \varphi^{[-1]}\left(\varphi(s) + \varphi(t)\right). \tag{3}$$

The function φ is called the *additive generator* of C.

In the work of Averous and Dortet–Bernadet[2] a nice relationship is established between the dependence properties of the copula (3) and the aging properties of the distribution function

$$F_\varphi(t) := 1 - \varphi^{[-1]}(t).$$

In the class of Archimedean copulas the most widely studied is the family of Frank's copulas[9]; if θ is a real parameter different from zero, then the Frank's family of copulas is defined through

$$C_\theta(s,t) := -\frac{1}{\theta}\ln\left(1 + \frac{\left(e^{-\theta s} - 1\right)\left(e^{-\theta t} - 1\right)}{e^{-\theta} - 1}\right).$$

The additive generator of C_θ is

$$\varphi_\theta(t) := -\ln\frac{e^{-\theta t} - 1}{e^{-\theta} - 1}.$$

It is important to note the following limiting values

$$\lim_{\theta\to-\infty} C_\theta = W, \qquad \lim_{\theta\to 0} C_\theta = \Pi, \qquad \lim_{\theta\to+\infty} C_\theta = M.$$

3. Statistical Properties

Copulæ are widely used in non–parametric statistics, especially in the study of dependence of random variables. For the notion of dependence see, for instance, the books by Szekli[28] and by Joe[13]. Many interesting known measures of association between random variables may be expressed in terms of copulas. If X and Y are continuous random variables and C is their copula, then Kendall's tau is

$$\tau_{X,Y} = 4\int_{[0,1]\times[0,1]} C(s,t)\,dC(s,t) - 1;$$

Spearman's rho is

$$\rho_{X,Y} = 12\int_{[0,1]\times[0,1]} st\,dC(s,t) - 3 = 12\int_{[0,1]\times[0,1]} C(s,t)\,ds\,dt - 3;$$

and Gini's measure of association is

$$\gamma_C = 4\int_0^1 C(t,1-t)\,dt - 4\int_0^1 \left(1 - C(t,t)\right)\,dt.$$

A measure of dependence was introduced by Schweizer and Wolff[22] in terms of the copula of the two random variables involved; its L^1-version is given by

$$\sigma_1(X,Y) := 12 \int_{[0,1]\times[0,1]} |C(s,t) - st|\, ds\, dt.$$

The L^p-version, with $p \in\,]1,+\infty[$, is given by

$$\sigma_p(X,Y) := \left\{ k_p \int_{[0,1]\times[0,1]} |C(s,t) - st|^p\, ds\, dt \right\}^{1/p};$$

here k_p is a suitable normalization factor, necessary for the inequality

$$0 \le \sigma_p(X,Y) \le 1.$$

For instance, in the case $p = 2$, one has $k_2 = 90$. There is also a L^∞-version given by

$$\sigma_\infty := 4 \sup\left\{ |C(s,t) - st| : s,t \in [0,1] \right\}.$$

All these measures of dependence meet a slight modification of Rényi's list of requirements for such a measure[18]; among other properties, Rényi requested that a measure of dependence $R(X,Y)$ of two random variables X and Y defined on a common probability space should satisfy $R(X,Y) = 1$ if either $X = f \circ X$ or $Y = g \circ X$ for some Borel-measurable functions f and g, and $R(f \circ X, g \circ Y) = R(X,Y)$ when f and g are one-to-one Borel-measurable functions. Instead the measures σ_p defined above are such that $\sigma_p(X,Y) = 1$ and $\sigma_p(f \circ X, g \circ Y) = \sigma_p(X,Y)$, if, and only if, f and g are strictly monotone.

Beside satisfying Rényi's requirements in the modified form just mentioned, the measures of dependence σ_p are such that if $\{(X_n, Y_n)\}$ converges weakly to (X,Y), then $\lim_{n\to+\infty} \sigma_p(X_n,Y_n) = \sigma(X,Y)$.

One can define *empirical copulas.* If $\{(X_j,Y_j) : j = 1,2,\dots,n\}$ is a sample of size n from a bivariate distribution and if the order statistics from the same sample are denoted by $X_{(j)}$ and by $Y_{(j)}$ then the empirical copula is defined via

$$C_n\left(\frac{j}{n}, \frac{k}{n}\right) := \frac{\text{number of pairs } (X,Y) \text{ with } X \le X_{(j)} \text{ and } Y \le Y_{(k)}}{n}.$$

Empirical copulæ were first used by Deheuvels[6] in order to construct tests of independence.

Recently copulæ have been used in Finance, see, for instance[10,8].

4. Copulæ and Markov Processes

Darsow, Nguyen and Olsen[5] established a connection between copulas and Markov processes through an operation on the set $\mathcal{C}$ of all copulas, which will now be described. Let A and B be copulæ; if x and y are in $[0,1]$ an operation on $\mathcal{C}$ is defined via

$$(A * B)(x,y) := \int_0^1 D_2 A(x,t)\, D_1 B(t,y)\, dt. \tag{4}$$

Then $A * B$ is a copula, the operation $*$ is associative, and, if $\{A_n\}$ converges to A, then one has both

$$A_n * B \xrightarrow[n \to +\infty]{} A * B$$

and

$$B * A_n \xrightarrow[n \to +\infty]{} B * A;$$

however, the operation $*$ is not jointly continuous. The operation $*$ has both a zero and an identity: they are, respectively, the copulæ Π and M; in fact, for every copula C, one has

$$\Pi * C = C * \Pi = \Pi,$$
$$M * C = C * M = C.$$

It follows from these relationships, and this remark will be important in the sequel, that both Π and M are idempotent, in the sense that

$$\Pi * \Pi = \Pi,$$
$$M * M = M.$$

The importance of the operation $*$ stems from the following

Theorem 2: (Darsow, Nguyen, Olsen) *Let $\{X_t : t \in T\}$ be a real–valued stochastic process and let C_{st} be the copula of the random variables X_s and X_t. If, for s and t in T with $s < t$ and for a Borel set A, one sets*

$$P(s,x,t,A) := P\left(X_t \in A \mid X_s = x\right),$$

then the following are equivalent:

(a) The Chapman–Kolmogorov equations

$$P(s,x,t,A) = \int_{\mathbf{R}} P(u,\xi,t,A)\, P(s,x,u,d\xi) \quad (u \in]s,t[\cap T)$$

hold for almost all $x \in \mathbf{R}$*;*

(b) $C_{st} = C_{su} * C_{ut}$.

The crucial step in the proof of Theorem 2 is the following equality

$$P(s,t,x,]-\infty,a]) = D_1 C_{st}\left(F_s(x), F_t(a)\right) \qquad \text{a.e..}$$

Of course $\{X_t\}$ may satisfy the Chapman–Kolmogorov equations without being a Markov process. A necessary and sufficient condition in terms of the copulas C_{st} is known. In order to state it, it is necessary to have recourse to the notion of n–dimensional copula[21,15], or, briefly n–copula, i.e. the function $C : [0,1]^n \to [0,1]$ that expresses an n–dimensional distribution function H in terms of its one–dimensional marginals F_1, F_1,... F_n,

$$H(x_1, x_2, \dots, x_n) = C\left(F_1(x_1), F_2(x_2), \dots, F_n(x_n)\right).$$

Let A be an m–copula and let B be an n–copula; then define $A \star B : [0,1]^{m+n-1} \to [0,1]$ via

$$(A \star B)(x_1, x_2, \dots, x_{m+n-1}) := \int_0^{x_m} D_m A(x_1, \dots, x_{m-1}, t)\, D_1 B(t, x_{m+1}, \dots, x_{m+n-1})\, dt.$$

Here again D_m denotes the partial derivative with respect to the m–th variable.

Then $A \star B$ is an $(m+n-1)$–copula and the operation $\star$ is associative, viz. $(A \star B) \star C = A \star (B \star C)$. The proclaimed necessary and sufficient condition is contained in the following theorem[5].

Theorem 3: *For a real–valued stochastic process* $\{X_t : t \in T\}$*, the following conditions are equivalent*

(a) $\{X_t : t \in T\}$ *is a Markov process;*

(b) for every natural number n *and for every choice of* n *elements* t_1, t_2, ..., t_n *in* T*, with* $t_1 < t_2 < \cdots < t_n$*, one has*

$$C_{t_1,t_2,\dots,t_n} = C_{t_1,t_2} \star C_{t_2,t_3} \star \cdots \star C_{t_{n-1},t_n},$$

where $C_{t_1,t_2,\dots,t_n}$ *is the copula of the random vector* $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ *and* C_{t_{k-1},t_k} *is the copula of the vector* $\left(X_{t_{k-1}}, X_{t_k}\right)$*, with* $k = 2, 3, \dots, n$*.*

5. Copulæ, Markov Operators and Conditional Expectations

While proving Theorem 2, Darsow, Nguyen and Olsen[5] proved that if the random variables X and Y have copula C, then one has almost surely

$$E\left(1_{\{X<x\}} \mid Y\right)(\omega) = D_2C\left(F_X(x), F_Y(Y(\omega))\right),$$
$$E\left(1_{\{Y<y\}} \mid x\right)(\omega) = D_1C\left(F_X(X(\omega)), F_Y(y)\right).$$

These latter relationships point at a connection between Conditional Expectations(=CE's) and copulas. This connection is best established through Markov operators. Given a probability space $(\Omega, \mathcal{F}, P)$, a *Markov operator* is a linear operator $T : L^\infty \to L^\infty$ such that

(a) T is positive, $f \geq 0 \implies Tf \geq 0$;
(b) $T1 = 1$;
(c) for every function f in L^∞, one has $E(Tf) = E(f)$.

Above, L^∞ may be replaced by L^1. Notice that if $\mathcal{G}$ is a sub–σ–field of $\mathcal{F}$, then the CE $E_\mathcal{G} := E(\cdot \mid \mathcal{G})$ is a Markov operator. It is then natural to ask which Markov operators are also CE's. The answer to this question is obtained by exploiting the expectation invariance of a Markov operator, the property expressed by (c) above and the characterization of CE's given by Pfanzagl[17] (but see also the previous works of Bahadur[3] and Šidák[24]); it is then possible to identify those Markov operators that are also CE's, when the probability space under consideration is $([0,1], \mathcal{B}, \lambda)$.

Theorem 4: *A Markov operator $T : L^\infty([0,1]) \to L^\infty([0,1])$ is the restriction to $L^\infty([0,1])$ of a CE if, and only if, it is idempotent, viz. $T^2 := T \circ T = T$. When this latter condition is satisfied, then $T = E_\mathcal{G}$, where $\mathcal{G} := \{A \in \mathcal{B} : T\,1_A = 1_A\}$.*

On the other hand, an explicit one–to–one correspondence between Markov operators and copulas can be established. It was shown[16] that, for every copula C, the operator T_C defined by

$$(T_C\, f)(x) := \frac{d}{dx} \int_0^1 D_2\, C(x,t)\, f(t)\, dt \tag{5}$$

is a Markov operator on $L^\infty([0,1])$ and that, conversely, if T is a Markov operator on $L^\infty([0,1])$, then the function $C_T : [0,1] \times [0,1] \to [0,1]$ defined by

$$C_T(x,y) := \int_0^x \left(T\, 1_{[0,y]}\right)(s)\, ds \tag{6}$$

is a copula. Thus, to a copula C there corresponds a unique Markov operator T_C and, conversely, to a Markov operator T there corresponds a unique copula C_T. Moreover, the composition of the two Markov operators T_A and T_B that correspond to the two copulas A and B is connected to the $*$ operation by means of the relationship

$$T_{A*B} = T_A \circ T_B.$$

Therefore the Markov operator T_C corresponding to a copula C is idempotent, and, hence a CE, if, and only if, the copula C is itself idempotent with respect to the operation $*$ introduced above, namely if, and only if, $C = C * C$. Thus, there exists a one–to–one correspondence between idempotent copulas and CE's in the probability space $([0,1], \mathcal{B}, \lambda)$; for this, see the author's paper[23].

The correspondence between Markov operators and copulas allows to establish a correspondence between copulas and measure preserving transformations on the unit interval[16,30]. We recall that a function $f : [0,1] \to [0,1]$ is said to be a *measure preserving transformation* if, for every Borel subset B of $\mathcal{B}$, $f^{-1}(B)$ is measurable, namely it belongs to $\mathcal{B}$, and one has

$$\lambda\left(f^{-1}(B)\right) = \lambda(B).$$

If f and g are measure preserving transformations, the function $C_{f,g}$: $[0,1]^2 \to [0,1]$ defined by

$$C_{f,g}(s,t) := \lambda\left(f^{-1}([0,s]) \cap g^{-1}([0,t])\right),$$

is a copula. Conversely, for every copula C there exists a pair of measure preserving transformations f and g, such that

$$C = C_{f,g}.$$

6. Generalizations of Copulas

In spite of their many uses in probability and statistics, uses which we have tried to sketch in the previous sections, it has been necessary to generalize the notion of copula in order to deal with certain problems. The first such generalization was introduced by Alsina, Nelsen and Schweizer[1] in order to characterize a class of operations on distribution functions that derive from corresponding operations on random variables defined on the same probability space. The concept of *track* is needed: a track is a subset B of the unit square $[0,1] \times [0,1]$ that can be written in the form

$$B = \{(F(t), G(t)) : t \in [0,1]\},$$

for some continuous distribution functions F and G such that $F(0) = G(0) = 0$ and $F(1) = G(1) = 1$. Then, a *quasi–copula* is a function $Q : [0,1] \times [0,1] \to [0,1]$ such that, for every track B, there exists a copula C_B that coincides with Q on the points of B: for all $(s,t) \in B$

$$Q(s,t) = C_B(s,t).$$

Later it was proved[11] that Q is a quasi–copula if, and only if, it meets the following requirements

(a) $\forall t \in [0,1] \qquad Q(0,t) = Q(t,0) = 0, \quad \text{and} \quad Q(1,t) = Q(t,1) = t;$
(b) the function $(s,t) \mapsto Q(s,t)$ is non-decreasing in each of its arguments;
(c) Q satisfies Lipschitz's condition, viz., for all s, s', t and t' in $[0,1]$,

$$|Q(s',t') - Q(s,t)| \leq |s' - s| + |t' - t|.$$

The properties of quasi–copulas have been extensively studied in Úbeda Flores's dissertation[29].

Bassan and Spizzichino[4] have introduced another generalization of the concept of copula in their investigation of bivariate aging. Let X and Y be two positive and exchangeable random variables and let $\overline{F}$ be their joint survival function

$$\overline{F}(s,t) := P(X > s, Y > t) \qquad s,t > 0.$$

Finally, let $\overline{G}$ be the univariate marginal survival function of $\overline{F}$, $\overline{G}(x) := \overline{F}(x,0) = \overline{F}(0,x)$. Then they introduce the *bivariate aging function* $B : [0,1]^2 \to [0,1]$ defined by

$$B(s,t) := \exp\left\{-\overline{G}^{[-1]}\left(\overline{F}\left(-\ln s, -\ln t\right)\right)\right\}.$$

This is an increasing function of each of its arguments, but is not necessarily a quasi–copula (nor, *a fortiori*, a copula).

A generalization in a different, more abstract, direction, has been introduced by Scarsini[19]; here, we shall briefly touch only on the case $n = 2$. Let $(\Omega_1, \mathcal{B}_1, P_1)$ and $(\Omega_2, \mathcal{B}_2, P_2)$ be probability spaces, where Ω_1 and Ω_2 are Polish (*i.e.* complete metrizable) spaces, each of them endowed with its Borel σ–field $\mathcal{B}_i$ $(i = 1,2)$. Let $\mathcal{A}_i$ be an increasing class of sets belonging to $\mathcal{B}_i$ $(i = 1,2)$ (if the sets A and B are in $\mathcal{A}_i$, then either $A \subset B$ or $B \subset A$, or $A = B$) and let $(\Omega_1 \times \Omega_2, \mathcal{B}_1 \otimes \mathcal{B}_2, \mu)$ be a probability space such that, for all $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$, one has

$$\mu(B_1 \times \Omega_2) = P_1(B_1), \qquad \text{and} \qquad \mu(\Omega_1 \times B_2) = P_2(B_2).$$

Then, if one sets

$$\mathcal{A} := \{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\},$$

there exists a copula $C_\mu^{\mathcal{A}}$ such that, for every choice of $A_1 \times A_2$ in $\mathcal{A}$, one has

$$\mu(A_1 \times A_2) = C_\mu^{\mathcal{A}}(P_1(A_1), P_2(A_2)).$$

References

1. C. Alsina, R.B. Nelsen, B. Schweizer, On the characterization of a class of binary operations on distribution functions, *Statistics & Probability Letters*, **17**, 85–89 (1993).
2. J. Averous, J.–L. Dortet–Bernadet, *Dependence for archimedean copulas and aging properties of their generating functions*, preprint, 2002, to appear.
3. R.R. Bahadur, Measurable subspaces and subalgebras, *Proceedings of the American Mathematical Society*, **6**, 565–570 (1955).
4. B. Bassan, F. Spizzichino, Dependence and multivariate aging: the role of level sets of the survival function, *Series on Quality and Reliability in Engineering and Statistics*, **5**, 229–242 (2001).
5. W.F. Darsow, B. Nguyen, E.T. Olsen, Copulas and Markov processes, *Illinois Journal of Mathematics*, **36**, 600-642 (1992).
6. P. Deheuvels, La fonction de dépendence empirique et ses propriétés. Un test non paramétrique d'indépendence, *Académie Royale de Belgique. Bulletin de la Classe de Sceinces*, (5) **65**, 274–292 (1979).
7. C. De Michele, G. Salvadori, A generalized Pareto intensity–duration model of storm rainfall exploiting 2–copulas, to appear
8. P. Embrechts, A. McNeil, D. Straudermann, Correlation and Dependence in Risk Management: Properties and Pitfalls, in *Risk Management: Value at Risk and Beyond*, Ed. M. Dempster (Cambridge University Press, 2002) pp. 176–223.
9. M.J. Frank, On the simoultaneous associativity of $F(x,y)$ and $x+y-F(x,y)$, *Aequationes Mathematicae*, **19**, 194–226 (1979).
10. E.W. Frees E. Valdez, Understanding Relationships using Copulas, *North American Actuarial Journal*, **2**, 1–25 (1998).
11. C. Genest, J.J. Quesada Molina, J.A. Rodríguez Lallena, C. Sempi, A characterization of quasi–copulas, *Journal of Multivariate Analysis*, **69**, 193–205 (1999).
12. T.P. Hutchinson, C.D. Lai, *Continuous bivariate distributions. Emphasising applications* (Rumsby Scientific Publishing, Adelaide, 1990).
13. H. Joe, *Multivariate models and dependence concepts* (Chapman & Hall, London, 1997).
14. P. Mikusiński, H. Sherwood, M.D. Taylor, The Fréchet bounds revisited, *Real Analysis Exchange*, **17**, 759–764 (1991-92).
15. R.B. Nelsen, *An Introduction to Copulas* (Springer, New York, 1999).